Constructing finite difference schemes.

Example: 4^{th} order, CIS finite difference scheme for the first derivative: f^i

Start with:

(1) $f_{m+2} = f_m + f^i_m\{2D/1!\} + f^{ii}_m\{4D^2/2!\} + f^{iii}_m\{8D^3/3!\} + f^{iv}_m\{16D^4/4!\} + f^v_m\{32D^5/5!\} + \dots$

(2) $f_{m+1} = f_m + f^i_m\{D/1!\} + f^{ii}_m\{D^2/2!\} + f^{iii}_m\{D^3/3!\} + f^{iv}_m\{D^4/4!\} + f^v_m\{D^5/5!\} + \dots$

(3) $f_{m-1} = f_m - f^i_m\{D/1!\} + f^{ii}_m\{D^2/2!\} - f^{iii}_m\{D^3/3!\} + f^{iv}_m\{D^4/4!\} - f^v_m\{D^5/5!\} + \dots$

(4) $f_{m-2} = f_m - f^i_m\{2D/1!\} + f^{ii}_m\{4D^2/2!\} - f^{iii}_m\{8D^3/3!\} + f^{iv}_m\{16D^4/4!\} - f^v_m\{32D^5/5!\} + \dots$

Consider the sum: P*(1) + Q*(2) + R*(3) + S*(4) as follows:

$P*[\ f_{m+2} = f_m + f^i_m\{2D/1!\} + f^{ii}_m\{4D^2/2!\} + f^{iii}_m\{8D^3/3!\} + f^{iv}_m\{16D^4/4!\} + f^v_m\{32D^5/5!\} + \dots]$

$Q*[f_{m+1} = f_m + f^i_m\{D/1!\} + f^{ii}_m\{D^2/2!\} + f^{iii}_m\{D^3/3!\} + f^{iv}_m\{D^4/4!\} + f^v_m\{D^5/5!\} + \dots\ ]$

$R*[f_{m-1} = f_m - f^i_m\{D/1!\} + f^{ii}_m\{D^2/2!\} - f^{iii}_m\{D^3/3!\} + f^{iv}_m\{D^4/4!\} - f^v_m\{D^5/5!\} + \dots\ ]$

$S*[f_{m-2} = f_m - f^i_m\{2D/1!\} + f^{ii}_m\{4D^2/2!\} - f^{iii}_m\{8D^3/3!\} + f^{iv}_m\{16D^4/4!\} - f^v_m\{32D^5/5!\} + \dots\ ]$

Summing up the columns above gives:

column on the LHS =

(5) $P* f_{m+2} + Q*f_{m+1} + R*f_{m-1} + S*f_{m-2}$

1^{st} column on RHS =

(6) $(P + Q + R + S)*f_m$

2^{nd} column on RHS =

(7) $f^i_m*\{D/1!\}*(2*P + Q - R - 2*S)$

3^{rd} column on RHS =

(8) $f^{ii}_m*\{D^2/2!\}*(4*P + Q + R + 4*S)$

4^{th} column on RHS =

(9) $f^{iii}_m*\{D^3/3!\}*(8*P + Q - R - 8*S)$

5^{th} column on RHS =

(10) $f^{iv}_m*\{D^4/4!\}*(16*P + Q + R + 16*S)$

6^{th} column on RHS =

(11) $f^v_m*\{D^5/5!\}*(32*P + Q - R - 32*S)$

To eliminate a term set the sum in the ( ) =0. for that column.

For a 4th order centered in space scheme, set the ( ) terms separately to zero in (8) – (10):

(12) $4*P + Q + R + 4*S = 0.$

(13) $8*P + Q - R - 8*S = 0.$

(14) $16*P + Q + R + 16*S = 0.$

Note that in this case we don't touch (11) because that will be the first term in the remainder.

And, set the ( ) terms in (7) = 1.

(15) $2*P + Q - R - 2*S = 1.$

Solve for P, Q, R, and S. Note that

(14) - (12) = $12*P + 12S = 0.$ → $P = - S$ (16)

Also,

4*(12)-(14) = $3*Q + 3*R =0.$ → $Q = - R$ (17)

Substitute (16) and (17) into the eqns not yet used: (13) and (15) to get:

$16*P + 2*Q = 0.$ → $8*P + Q = 0.$ (18)

$4*P + 2*Q = 1.$ → $2*P + Q = 1.$ (19)

Thus:

(18)-(19) = $6*P = -1$ → $P = -1./6.$ (20)

From (16) → $S = 1./6.$ (21)

From (18) $-8/6 + Q = 0.$ → $Q = 4./3.$ (22)

From (17) → $R = -4./3.$ (23)

Reconstructing the finite difference scheme, using (5)-(7):

f^i_m = { (5) - (6) }/ D

(24) $f^i_m = \{ P^* f_{m+2} + Q^*f_{m+1} + R^*f_{m-1} + S^*f_{m-2} - (P + Q + R + S)^*f_m \}/ D$

Substituting (20)-(23) yields the desired finite difference formula:

(25) $f^i_m = \{ -1/6\ f_{m+2} +4/3\ f_{m+1} -4/3\ f_{m-1} +1/6\ f_{m-2} \}/ D$

where it turns out that $P + Q + R + S = 0.$

Note this simple check: if you sum the coefficients up inside the { } in (25) they should and do equal zero.