Streamfunction Depiction of (Zonal) Mean Meridional Circulations (MMC)

define streamfunction $\psi \equiv \frac{2\pi r \cos\phi}{g} \int_0^p \overline{[v]}\, dp$ (1)

note that (1) is an indefinite integral. The variables have their usual meanings. Ditto for symbols.

$$\frac{\partial}{\partial p}(1) \Rightarrow \quad \frac{\partial \psi}{\partial p} = \frac{2\pi r \cos\phi}{g} \left\{ \overline{[v(p)]} - \underbrace{\cancel{\overline{[v(0)]}}}_{0} \right\}$$

rearranging:

$$\overline{[v(p)]} = \frac{g}{2\pi r \cos\phi} \frac{\partial \psi(p)}{\partial p} \qquad \text{"(3.5a)"}$$

Obtain $\overline{[\omega]}$ from continuity equ. in spherical coor:

$$\frac{\partial \omega}{\partial p} + \frac{1}{r\cos\phi} \left\{ \frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \phi}\left(v \cos\phi\right) \right\} = 0$$

take zonal average: [ ]. Assume that average does not intersect any mountain ranges so that $\left[\frac{\partial u}{\partial \lambda}\right] = 0$

So:

$$\left[\frac{\partial \omega}{\partial p}\right] + \frac{1}{r\cos\phi} \left[\frac{\partial}{\partial \phi}\left(v \cos\phi\right)\right] = 0 \qquad (2)$$

Sub (3.5a) into (2) to obtain, after rearranging:

$$\left[\frac{\partial \omega}{\partial p}\right] = \frac{-1}{r\cos\phi} \frac{\partial}{\partial \phi}\left(\overline{[v]} \cos\phi\right)$$

Since ϕ is not a function of λ.

$$\frac{\partial}{\partial p}[\omega] = \frac{-1}{r\cos\phi} \frac{\partial}{\partial \phi}\left(\frac{g \cos\phi}{2\pi r \cos\phi} \frac{\partial \psi}{\partial p} \right)$$

neither is P

Integrate from top of atmos. ($p=0$) to some level p:

$$\int_0^p \frac{\partial}{\partial p}[\omega]\,dp = [\omega]\Big|_p - \underbrace{\cancel{[\omega]}\Big|_{p=0}}_{0} = \frac{-g}{2\pi r^2\cos\varphi}\int_0^p \frac{\partial^2\psi}{\partial p\,\partial\varphi}\,dp$$

$$[\omega(p)] = \frac{-g}{2\pi r^2\cos\varphi}\left\{ \frac{\partial\psi}{\partial\varphi}\Big|_{p=p} - \underbrace{\cancel{\frac{\partial\psi}{\partial\varphi}}\Big|_{p=0}}_{0} \right\}$$

last term neglected if $\psi = 0$ at $p = 0$. Thus, taking time average:

$$\overline{[\omega]} = \frac{-g}{2\pi r^2\cos\varphi}\,\frac{\partial\psi}{\partial\varphi} \qquad \text{"(3.5b)"}$$

Notice that flow is 2-D, since continuity eqn using just 2 terms was used to obtain $[\omega]$ from $[v]$.

So far assumed: $[v]=0$ & $\psi =$ constant, 0 say, at $p=0$.

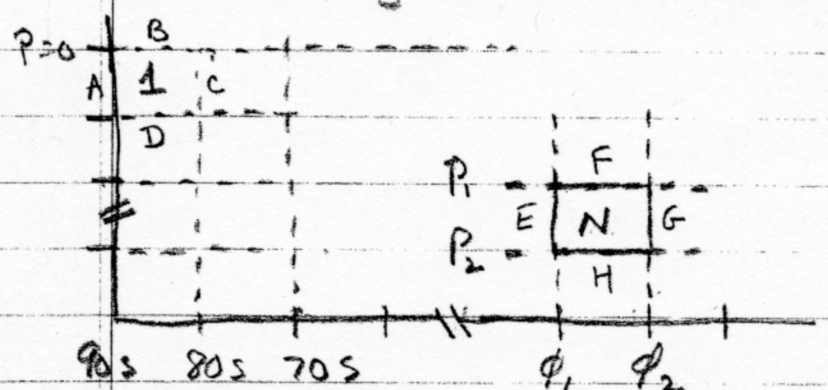
Procedure is to obtain $[v]$ at grid points in meridional plane then use (1) to obtain ψ. The ψ field will show $[\omega]$ & $[v]$ in 1 scalar field.

why $\overline{[v]}$?

1. direct measurement feasible in tropics (not $[\omega]$)
2. indirect technique can estimate $[v]$ and $[\omega]$ but not in tropics

Indirect method

1. divide meridional cross section into rectangular areas
2. Obtain average velocity normal to each side.
3. The velocities are thus "grid point" values
4. Use conservation of mass and of angular momentum
5. Close system with various assumptions and by exploiting boundary conditions



Box 1 is the 1st rectangle to be done

Box N is arbitrary example

Angular Momentum (M) formula (P.91)

$$M = R^2 \Omega + R u \qquad R = r \cos\phi$$

neglecting by assumption:

a. mountain torques; b. friction; c. time tendency

Then what enters box = what leaves box. For box N:

$$\int_{S_E} R_E^2 \Omega v_E \, ds + \int_{S_E} R_E v_E u_E \, ds + \int_{S_F} R_F^2 \Omega \omega_F \, ds + \int_{S_F} R_F \omega_F u_F \, ds$$

$$= \int_{S_G} R_G^2 \Omega v_G \, ds + \int_{S_G} R_G v_G u_G \, ds + \int_{S_H} R_H^2 \Omega \omega_H \, ds + \int_{S_H} R_H \omega_H u_H \, ds \qquad (3)$$

Let: A_E is $2\pi r \cos\phi_1 \times (P_2 - P_1)$ surface area that integral $\int_{S_E}$ is over.

A_F is $r^2 2\pi \int_{\phi_1}^{\phi_2} \cos\phi\, d\phi$ surface area for $\int_{S_F}$.

etc. for A_G and A_H.

Next assume; neglect ωu terms.

Note also that since these are averages, the velocities can be brought outside each integral.

Defining $\hat{R}_H = \frac{1}{A_H}\int_{S_H} R\, ds$ and $\hat{R}_F = \frac{1}{A_F}\int_{S_F} R\, ds$

Ignore variation of R in vertical surfaces: R_E, R_G constants

Rearranging, then (3) becomes:

$$A_E R_E^2 \Omega [v_E] + A_E R_E [v_E u_E] + A_F \hat{R}_F^2 \Omega [\omega_F] =$$

$$(4) \qquad A_G R_G^2 \Omega [v_G] + A_G R_G [v_G u_G] + A_H \hat{R}_H^2 \Omega [\omega_H]$$

Apply (4) for box 1:

$[v_A] = 0$, $[\omega_B] = 0$, $[v_A u_A] = 0$ from boundaries at pole, top

Equate mass fluxes through remaining 2 sides:

$$\int_{S_D} [\omega_D]\, r\cos\phi\, ds \simeq \hat{R}_D [\omega_D]\, 2\pi \Delta\phi\, r$$

$$(5) \qquad = R_C [v_C]\, 2\pi \Delta P \simeq \int_{S_C} [v_C]\, r\cos\phi\, ds$$

where $\Delta\phi = \phi_2 - \phi_1$, $\Delta P = P_2 - P_1$

Simplifying: $$[\omega_D] = \frac{R_C \Delta P}{\hat{R}_D r \Delta\phi} [v_C] \qquad (6)$$

subbing (6) into (4); rearrange:

$$0 = A_C R_C^2 \Omega [v_C] + A_C R_C [v_C u_C] + \frac{R_C \hat{R}_D^2}{\hat{R}_D} \frac{\Delta P A_D}{r \Delta\phi} \Omega [v_C]$$

Solving for $$[v_C] = \frac{-A_C R_C [v_C u_C]}{A_C R_C^2 \Omega + \frac{R_C R_D}{r} \frac{\Delta P}{\Delta\phi} \Omega A_D} \qquad (7)$$

(7) provides $[\bar{v}]$ on face labelled C.
Sub result into (6) to get $[\omega]$ on face labelled D.
Now do next box below.
$[v]$ from pole is still 0, $[\omega]$ at top is not zero but is known. Obtain $[v]$ and $[\omega]$ similar to formulae (7) and (6)
Similarly can move towards next box at higher latitude. By moving to adjacent boxes you always know normal velocity on 2 sides of the box and obtain the other 2 sides using similar equations as (7) and (6).
In general (by inspection of (4)

$$\nu\left(A_H \hat{R}_H^2 [\omega_H] + A_G R_G^2 [v_G]\right) = A_E R_E^2 \nu [v_E] + A_E R_E [v_E u_E] + A_F \hat{R}_F^2 \nu [\omega_F] - A_G R_G [v_G u_G] = RHS \qquad (8)$$

everything in RHS is known.
To eliminate $[\omega_H]$ again equate mass fluxes:
Using (5) as a guide:

$$[\omega_H] = \frac{1}{2\pi r \Delta\phi \hat{R}_H} \left\{ R_G [v_G] 2\pi \Delta P - \underbrace{R_E [v_E] 2\pi \Delta P - 2\pi r \Delta\phi \hat{R}_F [\omega_F]}_{\text{known, put in RHS of (8)}} \right\} \qquad (9)$$

Combine (8) and (9) to obtain formula like (7) for $[v_G]$
Once $[v_G]$ known, sub into (9) to get $[\omega_H]$, then move on to the next block.
Do same thing working equatorward from N. Pole.
In subtropics, switch to directly measured $[v]$
Once $[v]$ at all grid points, final step is to use (1) to get ψ.