Appendix C

Useful Equations

Purposes: Provide foundation equations and sketch some derivations. These equations are used as starting places for discussions in various parts of the book.

C.1. Thermodynamic equations

Thermodynamics is important in understanding the general circulation. Several key equations are diagnostic. The ideal gas law (C.1) often used to relate mass and temperature variables:

$$p = \rho RT \tag{C.1}$$

Temperature is not conserved when moving adiabatically while undergoing compression or expansion, but a related variable, potential temperature, θ is conserved. Poisson's equation (C.2) relates the two:

$$\theta = T\left(\frac{p_{oo}}{p}\right)^{R/C_P} \tag{C.2}$$

where p_{oo} is a reference level constant pressure (often 10^5 Pa). Since θ is conserved for adiabatic motions, a prognostic temperature equation is:

$$\frac{d\theta}{dt} = \dot{\theta} = D_J \frac{\theta}{C_p T} = D_T \frac{\theta}{T} \tag{C.3}$$

where D_J contains all the diabatic heating with units J s^{-1} kg^{-1} and D_T is a corresponding heating rate with units K/s. The first law of thermodynamics relates how heat exchanges, including work done on or by the environment, in order to conserve energy. An increment of such work per unit mass, dw could be written:
$dw = p\, d\alpha$

This dw can enter a meteorological problem as the change in temperature due to moving a mass (air parcel) vertically (and adiabatically) so that the mass is compressed or expanded. An increment in the non-adiabatic heating could be written as D_T. It can be written several ways (e.g. Wallace and Hobbs, 2006). One form, uses (C.1), (C.2) and (C.3):

$$\begin{aligned} &C_p \frac{d\ln T}{dt} - R\frac{d\ln p}{dt} = C_p \frac{d\ln\theta}{dt} = \frac{D_J}{T} \equiv \frac{dS_o}{dt} \\ &C_p \frac{dT}{dt} - \alpha\frac{dp}{dt} = D_J \end{aligned} \tag{C.4}$$

where S_o is entropy per unit mass (Holton, 2004) and from (C.4) d $S_o=C_p\,\mathrm{dln}(\theta)$. A related quantity, the enthalpy ($S_a=C_pT$) measures the heat added to raise the temperature of a mass from 0 K to T.

One major source of diabatic heating occurs when moisture has a net change of state. The amount of moisture available for a given temperature is related by the Clausius-Clapeyron equation:

$$\frac{de_s}{dT} = \frac{L}{T\alpha_v} \qquad \text{(C.5)}$$

where e_s is the saturation vapor pressure, and α_v is the specific volume for water vapor. (C.5) assumes that the specific volume for liquid water is much less than for vapor (Tsonis, 2007). Assuming that water vapor behaves like an ideal gas and employing Dalton's law of partial pressures, (C.5) can be rearranged as:

$$\frac{de_s}{e_s} = \frac{LdT}{R_v T^2} \qquad \text{(C.6)}$$

(C.6) shows that e_s is a nonlinear function of temperature. Assuming L is independent of T (L varies by ~10% over 273K ≤ T ≤ 373K) and integrating (C.6) from a reference temperature T_o to T obtains (James, 1994):

$$e_s(T) = e_s(T_o)\exp\left(\frac{L}{R_v}\left\{\frac{1}{T_o} - \frac{1}{T}\right\}\right) \qquad \text{(C.7)}$$

The dependence is approximately exponential. For $T>T_o$ the saturation vapor pressure increases as T increases. Alternatively, (Tsonis, 2007) for T_o=273K:

$$e_s(T) = 6.11\exp\left(19.83 - \frac{5417}{T}\right) \qquad \text{in hPa} \qquad \text{(C.8)}$$

Diabatic heating from net changes in the state of water can be included in a conserved temperature quantity, equivalent potential temperature, θ_e

$$\theta_e = \theta\exp\left(\frac{Lw_d}{C_p T}\right) \qquad \text{(C.9)}$$

where w_d is the mixing ratio (i.e. the saturation mixing ratio with respect to liquid water at the dewpoint temperature) and the specific heat of dry air (C_p) is assumed much larger than the specific heat from the fraction of water vapor present. Note that .

From the LHS of (C.4) one can relate the heating rate to the dry static energy (DSE) as:

$$\frac{dDSE}{dt} \equiv \frac{d(C_p T + \Phi)}{dt} = D_J \qquad \text{(C.10)}$$

where Φ is the geopotential. The DSE is the internal plus gravitational potential energy. Similarly, inspired by (C.9) with $w_d \approx q$ one can define a moist static energy (MSE) equation from (C.4), where MSE is conserved if the only diabatic processes are moist diabatic processes (so-called 'psuedoadiabatic' processes).

$$\frac{dMSE}{dt} \equiv \frac{d(C_p T + \Phi + Lq)}{dt} = D_{NM} \qquad \text{(C.11)}$$

where D_{NM} is diabatic heating from non-moisture sources (such as radiation and conduction) and q is the specific humidity.

A balance of forces between gravity and vertical pressure gradient force leads to an expression relating pressure variation with elevation. The resultant hydrostatic balance (C.13) follows from ignoring vertical accelerations and is a good approximation for large scale motions and is useful when converting between height and pressure coordinates.

$$\frac{\partial p}{\partial z} = -\rho g \qquad \text{(C.12)}$$

Other forms of hydrostatic balance, in pressure and isentropic coordinates can be deduced from the PGCF terms in (C.24) and (C.25) below.

Substituting (C.1) into (C.12) and rearranging obtains a variation on hydrostatic balance: $\partial p / p = -(RT / g)\partial z \equiv -\partial z / H$. Assuming a vertically-averaged temperature, this variation can be easily integrated to obtain an approximate formula wherein pressure is an exponential function of elevation: $p = p_{oo} \exp(-z / H)$. H has units of length, is an e-folding scale, and is called the scale height. A similar procedure can eliminate pressure to obtain an exponential function for density variation with elevation.

Combining (C.12) and (C.1) yields the hypsometric equation relating the thickness ΔZ between pressure surfaces p_t and p_b and the temperature of that layer

$$\Delta Z = Z_t - Z_b = \frac{R}{g}\int_{p_t}^{p_b} T d \ln p \qquad \text{(C.13)}$$

where a constant acceleration of gravity, g was assumed. Combining (C.12) and the definition of DSE (C.10) obtains the dry adiabatic lapse rate of temperature:

$$\Gamma_d \equiv -\frac{dT}{dz} = \frac{g}{C_p} \qquad \text{(C.14)}$$

A moist adiabatic lapse rate, Γ_m can be derived (Tsonis, 2007) from the first law of thermodynamics (C.4) and using hydrostatic balance (C.12).

$$\Gamma_m \equiv -\frac{dT}{dz} = \frac{g}{C_p}\left\{\frac{1+\dfrac{Lw}{RT}}{1+\dfrac{L^2 w}{C_p R_v T^2}}\right\} \qquad \text{(C.15)}$$

$\Gamma_m \le \Gamma_d$ since the term inside the brackets is less than one.

C.2. Equations of motion and continuity

The equations of motion and continuity are used in several different coordinate frames depending upon the problem being examined. The equations of motion, or 'momentum equations' (per unit mass) include a total derivative. The vector curl operation also appears in the momentum equations. The equation for mass continuity, or 'continuity equation' uses a divergence operator. These three operators differ with coordinate frame and so the general forms of the operators are discussed first, drawing inspiration from Haltiner and Williams (1980).

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \vec{V} \bullet \vec{\nabla} A = \frac{\partial A}{\partial t} + \frac{v_1}{m_1}\frac{\partial A}{\partial x_1} + \frac{v_2}{m_2}\frac{\partial A}{\partial x_2} + \frac{v_3}{m_3}\frac{\partial A}{\partial x_3} \qquad \text{(C.16)}$$

where A is a scalar. The three dimensional velocity vector $\vec{V} = (v_1, v_2, v_3)$ while the m_i is a map factor in the direction x_i.

A curl operator is defined as:

$$\vec{\nabla} \times \vec{G} = \frac{\vec{i}}{m_2 m_3}\left(\frac{\partial m_3 G_3}{\partial x_2} - \frac{\partial m_2 G_2}{\partial x_3}\right) + \frac{\vec{j}}{m_1 m_3}\left(\frac{\partial m_1 G_1}{\partial x_3} - \frac{\partial m_3 G_3}{\partial x_1}\right) + \frac{\vec{k}}{m_1 m_2}\left(\frac{\partial m_2 G_2}{\partial x_1} - \frac{\partial m_1 G_1}{\partial x_2}\right) \text{(C.17)}$$

where $\vec{G}$ =(G_1, G_2 ,G_3) is a vector. The divergence of $\vec{G}$ is:

$$\vec{\nabla} \bullet \vec{G} = \frac{1}{m_1 m_2 m_3}\left(\frac{\partial m_2 m_3 G_1}{\partial x_1} + \frac{\partial m_1 m_3 G_2}{\partial x_2} + \frac{\partial m_1 m_2 G_3}{\partial x_3}\right) \qquad \text{(C.18)}$$

Two commonly-used horizontal coordinate frames are Cartesian and spherical coordinates. The map factors for those systems are:

$$\begin{array}{ll} (x_1, x_2) = (x, y) & \text{Cartesian Coordinates} \\ (m_1, m_2, m_3) = (1,1,1) & \text{Cartesian Coordinates} \\ (n_1, n_2, n_3) = (0,0,0) & \text{Cartesian Coordinates} \\ (v_1, v_2) = (u, v) & \text{Cartesian Coordinates} \end{array} \qquad \text{(C.19)}$$

$$\begin{array}{ll} (x_1, x_2) = (\lambda, \varphi) & \text{Spherical Coordinates} \\ (m_1, m_2, m_3) = (r\cos\varphi, r, 1) & \text{Spherical Coordinates} \\ (n_1, n_2, n_3) = (1,1,1) & \text{Spherical Coordinates} \\ (v_1, v_2) = (r\cos\varphi \dfrac{d\lambda}{dt}, r\dfrac{d\varphi}{dt}) & \text{Spherical Coordinates} \end{array} \qquad \text{(C.20)}$$

where n_1, n_2, and n_3 are used below. In general, the wind components are:

$$(v_1, v_2, v_3) = \left(m_1 \frac{dx_1}{dt}, m_2 \frac{dx_2}{dt}, m_3 \frac{dx_3}{dt} \right) \qquad \text{(C.21)}$$

Several choices are commonly-used for the vertical coordinate, x_3 and the choice affects the definitions of the vertical derivatives and velocity component v_3. Specifically:

$$\begin{array}{ll} x_3 = z & \text{Height Coordinates} \\ v_3 = w & \text{Height Coordinates} \\ \left(n_o, n_t, n_z, n_v\right) = (\rho, 1, 1, 1) & \text{Height Coordinates} \\ h = p & \text{Height Coordinates} \\ PGCF = \dfrac{1}{\rho}\dfrac{\partial p}{\partial z} + \dfrac{\partial \Phi}{\partial z} = \dfrac{1}{\rho}\dfrac{\partial p}{\partial z} + g & \text{Height Coordinates} \end{array} \qquad \text{(C.22)}$$

$$\begin{array}{ll} x_3 = p & \text{Pressure Coordinates} \\ v_3 = \omega & \text{Pressure Coordinates} \\ \left(n_o, n_t, n_z, n_v\right) = \left(1, 1, \dfrac{-1}{\rho g}, \dfrac{-1}{\rho g}\right) & \text{Pressure Coordinates} \\ h = \Phi & \text{Pressure Coordinates} \\ PGCF = -g\left\{1 + \rho \dfrac{\partial \Phi}{\partial p}\right\} & \text{Pressure Coordinates} \end{array} \qquad \text{(C.23)}$$

$$x_3 = \theta \qquad \theta \text{ Coordinates}$$
$$v_3 = \dot{\theta} \qquad \theta \text{ Coordinates}$$
$$\left(n_o, n_t, n_z, n_v\right) = \left(1, \frac{\partial z}{\partial \theta}, 1, \frac{\partial z}{\partial \theta}\right) \qquad \theta \text{ Coordinates} \tag{C.24}$$
$$h = C_p T + \Phi \qquad \theta \text{ Coordinates}$$
$$PGCF = \frac{\partial \theta}{\partial z}\left\{\frac{\partial h}{\partial \theta} - \frac{C_p T}{\theta}\right\} = \frac{\partial \theta}{\partial z}\left\{\frac{\partial}{\partial \theta}\left(C_p T + \Phi\right) - \frac{C_p T}{\theta}\right\} \qquad \theta \text{ Coordinates}$$

where the h and n_o values are used in the equations below. The quantity h in potential temperature (‘theta’) coordinates is generally referred to as the Montgomery Stream function and it is obviously related to the DSE.

The so-called primitive equations of motion actually result from several assumptions applied to the Navier-Stokes equations, along with additional thermodynamic relations to have sufficient independent equations to match the number of dependent variables. Derivations can be found in several books, e.g. Vallis (2006).

The primitive equations include momentum equations which equate the acceleration of the wind components to operant body and surface forces. For large scale atmospheric flows gravity is the only body force while pressure gradient and friction (including diffusion) are surface forces. Apparent forces, like the Coriolis terms arise from a coordinate frame of reference rotating with respect to a fixed point on the Earth’s surface. (The Coriolis is sometimes thought of as a body force.)

Even the primitive equations form has three primary ‘traditional’ assumptions: 1) thin atmosphere (use a constant distance, r in (C.20) instead of the distance from the point in the atmosphere to the center of the earth), 2) ignore small ‘metric’ terms (uw/r and uw/r), 3) ignore Coriolis terms involving the vertical component of the wind.

Momentum is altered following the motion of an air parcel by body and surface forces and for the atmosphere can be written in vector form as:

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\Phi - 2\vec{\Omega}\times\vec{V} + \vec{F} \tag{C.25}$$

where the geopotential is defined to include both the gravitational geopotential and the centrifugal acceleration (due to the Earth’s rotation). It is understood that (C.25) is an equation for momentum per unit mass. Examining (C.25) it is clear that the pressure gradient term using height coordinates will not have a contribution from the $\vec{\nabla}\Phi$ term except for gravity in the vertical component equation. And, in pressure coordinates the $\vec{\nabla}p$ term vanishes except for the vertical component equation. For some other coordinate, like θ both terms are present.

The component momentum equations are:

$$\frac{dv_1}{dt} = -\frac{1}{n_o m_1}\frac{\partial h}{\partial x_1} + \left(2\Omega + \frac{n_1}{m_1}v_1\right)\left(v_2 \sin\varphi - n_3 n_v v_3 \cos\varphi\right) + F_1 \tag{C.26}$$

$$\frac{dv_2}{dt} = -\frac{1}{n_o m_2}\frac{\partial h}{\partial x_2} - \left(2\Omega + \frac{n_1}{m_1}v_1\right)v_1 \sin\varphi - \frac{n_2 n_v}{m_2}v_2 v_3 + F_2 \qquad \text{(C.27)}$$

$$\begin{aligned} n_z \frac{dn_t v_3}{dt} &= -\frac{1}{\rho m_3}\frac{\partial p}{\partial x_3}\frac{\partial x_3}{\partial z} - \frac{1}{m_3}\frac{\partial \Phi}{\partial x_3}\frac{\partial x_3}{\partial z} + 2\Omega v_1 \cos\varphi + \frac{n_2}{m_2}\left(v_1^2 + v_2^2\right) + F_3 \\ &= -PGCF + 2\Omega v_1 \cos\varphi + \frac{n_2}{m_2}\left(v_1^2 + v_2^2\right) + F_3 \end{aligned} \qquad \text{(C.28)}$$

The hydrostatic equation (C.12) is easily seen by retaining only the ‘pressure gradient’ and gravity forces in (C.28) which are contained in the term labeled ‘PGCF’ defined in (C.22). PGCF reveals the hydrostatic relation in p and θ coordinates in (C.23) and (C.24), respectively. The continuity equation is:

$$\frac{d}{dt}\left(\ln\left(\frac{\partial p}{\partial x_3}\right)\right) + \vec{\nabla} \bullet \vec{V} = 0 \qquad \text{(C.29)}$$

It is obvious from (C.29) and (C.23) that the continuity equation has a very simple form in pressure coordinates:

$$\vec{\nabla} \bullet \vec{V} = 0 \qquad \text{(C.30)}$$

Following Vallis (2006) (C.25) can be expressed using three dimensional vorticity $\vec{\zeta} = \vec{\nabla} \times \vec{V}$ and kinetic energy, $\frac{1}{2}\left(\vec{V} \bullet \vec{V}\right)$

$$\frac{\partial \vec{V}}{\partial t} = -n_o \vec{\nabla} h + \vec{g} - \left(2\vec{\Omega} \times \vec{\zeta}\right) \times \vec{V} - \frac{1}{2}\vec{\nabla}\left(\vec{V} \bullet \vec{V}\right) \qquad \text{(C.31)}$$

where $\vec{g}$ =(0,0,g) is the gravity vector and $\vec{F}$ is the friction vector.Using just the vertical component of the vorticity, ζ the horizontal momentum equations can be written

$$\frac{\partial \vec{V}_H}{\partial t} + \frac{v_3}{m_3}\frac{\partial \vec{V}_H}{\partial x_3} = -n_o \vec{\nabla}_H h - \left\{ f \times \zeta \right\} \vec{k} \times \vec{V}_H - \frac{1}{2}\vec{\nabla}\left(\vec{V}_H \bullet \vec{V}_H\right) \qquad \text{(C.32)}$$

where $\vec{V}_H$ = (v_1, v_2, 0) and f (=2Ωsinφ) is the Coriolis parameter and the absolute vorticity is inside the curly brackets.

C.3. Two simplifications

Two simplifications of these equations are noteworthy for this book: the shallow water system and the quasi-geostrophic system.

Either by formally integrating in the vertical or by considering only density and velocity fields that are independent of the vertical coordinate, one may obtain the shallow water equations (SWE) system of equations. Hydrostatic balance (C.12) is assumed so the pressure gradient force is independent of elevation. Considering (C.26), (C.27), and (C.29), constant density leads to linear pressure change with height and scaling eliminates some smaller Coriolis terms. The SWE simplify to:

$$\frac{d\tilde{v}_1}{dt} = -n_o \frac{\partial \tilde{h}}{\partial x_1} + f\,\tilde{v}_2 + \tilde{F}_1 \qquad \text{(C.33)}$$

$$\frac{d\tilde{v}_2}{dt} = -n_o \frac{\partial \tilde{h}}{\partial x_2} - f\,\tilde{v}_1 + \tilde{F}_2 \qquad \text{(C.34)}$$

$$\frac{d\tilde{h}}{dt}+\tilde{h}\vec{\nabla}\bullet\vec{V}=0 \qquad \text{(C.35)}$$

where tilda overbars indicate vertical average velocity and friction while $\tilde{h}$ is specifically the thickness of the fluid layer. Vertical variation may be created by applying the SW system to multiple layers. Topography and variable layer top allow $\tilde{h}$ to vary in the horizontal.

These equations have several advantages: they contain a wide variety of wave types found in the atmosphere and they have a simple and intuitive expression for potential vorticity.

Another system of interest is the quasi-geostrophic (QG) system. The derivation proceeds as follows. The momentum and thermodynamic equations, an example of the latter being (C.3), are nondimensionalized, typically using scales relevant to midlatitude frontal cyclones. Nondimensional weights are expressed in terms of the Rossby number (R_o) and different orders of the small R_o. The lowest order is geostrophic winds and a hydrostatic static state for the horizontally mean mass and temperature fields. Geostrophic winds balance the pressure gradient and largest Coriolis terms (usually assuming a constant value for f):

$$v_{g1}=-\frac{1}{n_o f_o}\frac{\partial h}{\partial x_2} \qquad \text{(C.36)}$$

$$v_{g2}=\frac{1}{n_o f_o}\frac{\partial h}{\partial x_1} \qquad \text{(C.37)}$$

where the subscript ‘g’ indicates these are geostrophic balance winds and f_o is a specified constant value of f.

A thermal wind shear relation can be derived by eliminating h from (C.36) and (C.37) while using the hydrostatic and ideal gas relations. Generally, a vertical shear in the horizontal wind will be proportional to a horizontal temperature gradient normal to the wind direction. For example, in pressure coordinates (Holton, 2004)

$$\frac{\partial\vec{V}_g}{\partial\ln p}=-\frac{R}{f}\vec{k}\times\vec{\nabla}_p T \qquad \text{(C.38)}$$

In Cartesian geometry, component form:

$$\frac{\partial u_g}{\partial p}=\frac{R}{pf}\left(\frac{\partial T}{\partial y}\right)_p \qquad \text{(C.39)}$$

$$\frac{\partial v_g}{\partial p}=-\frac{R}{pf}\left(\frac{\partial T}{\partial x}\right)_p \qquad \text{(C.40)}$$

where p subscript means the derivative is on a surface holding p constant.

The QG system is the next higher order in the small R_o number expansion. For latitudes other than the deep tropics, and for large scale motions above the planetary boundary layer the geostrophic component of the wind is generally much larger than the remaining non-geostrophic (‘ageostrophic’) part of the total wind. The small parameter expansion (e.g. Grotjahn, 1979) results in ageostrophic winds substituting for the advecting winds in a total derivative and for the advected quantity in the horizontal equations of motion. Hydrostatic balance applies to perturbations from the zonal mean mass fields. Hence,

$$\frac{d_g v_{g1}}{dt} = \frac{\partial v_{g1}}{\partial t} + \vec{V}_g \bullet \vec{\nabla} v_{g1} = \frac{\partial v_{g1}}{\partial t} + \frac{v_{g1}}{m_1}\frac{\partial v_{g1}}{\partial x_1} + \frac{v_{g2}}{m_2}\frac{\partial v_{g1}}{\partial x_2} = f v_{ag2} + F_1 \quad \text{(C.41)}$$

$$\frac{d_g v_{g2}}{dt} = \frac{\partial v_{g2}}{\partial t} + \vec{V}_g \bullet \vec{\nabla} v_{g2} = \frac{\partial v_{g2}}{\partial t} + \frac{v_{g2}}{m_1}\frac{\partial v_{g2}}{\partial x_1} + \frac{v_{g2}}{m_2}\frac{\partial v_{g2}}{\partial x_2} = -f v_{ag1} + F_2 \quad \text{(C.42)}$$

where the subscript ‘ag’ indicates the ageostrophic wind. Vertical motion is small relative to horizontal motions, even after accounting for the smaller vertical to horizontal scales aspect ratio. Vertical motion does not appear explicitly except through the divergence of those ageostrophic winds. Vertical motion is missing from (C.41) and (C.42) due to strong static stability, but must be included in a temperature tendency equation, for example:

$$\frac{d_g \theta}{dt} = \frac{\partial \theta}{\partial t} + \vec{V}_g \bullet \vec{\nabla}\theta + v_3 \frac{\partial \theta_s}{\partial x_3} = \frac{\partial \theta}{\partial t} + \frac{v_{g2}}{m_1}\frac{\partial \theta}{\partial x_1} + \frac{v_{g2}}{m_2}\frac{\partial \theta}{\partial x_2} + \frac{v_3}{m_3}\frac{\partial \theta_s}{\partial x_3} = \dot{\theta} = Q\frac{\theta_s}{T_s} \quad \text{(C.43)}$$

where the s subscripts refer to the horizontally uniform static state. Hydrostatic balance and (lower order) continuity equation complete the QG system.

C.3. Potential vorticity relations

A primary diagnostic is potential vorticity. It takes different mathematical forms in different vertical coordinates and systems of equations.

The form in Cartesian geometry of the SWE is particularly simple and intuitive. Forming a vorticity equation from y derivative of (C.33) and an x derivative of (C.34) while using (C.19) and (C.22) yields the inviscid form:

$$\begin{aligned}&\frac{d_g\left(\tilde{\zeta}_g + f\right)}{dt} = \frac{\partial \tilde{\zeta}_g}{\partial t} + \vec{V}_g \bullet \vec{\nabla}\tilde{\zeta}_g + \frac{\tilde{v}_{g2}}{m_2}\frac{\partial f}{\partial x_2} = \\ &\frac{\partial \tilde{\zeta}_g}{\partial t} + \frac{\tilde{v}_{g2}}{m_1}\frac{\partial \tilde{\zeta}_g}{\partial x_1} + \frac{\tilde{v}_{g2}}{m_2}\frac{\partial\left(\tilde{\zeta}_g + f\right)}{\partial x_2} = -f_o\left(\frac{1}{m_1}\frac{\partial \tilde{v}_1}{\partial x_1} + \frac{1}{m_2}\frac{\partial \tilde{v}_2}{\partial x_2}\right)\end{aligned} \quad \text{(C.44)}$$

Combining this equation with (C.35) to eliminate the divergence yields:

$$\frac{dQ_{SWE}}{dt} \equiv \frac{d}{dt}\left(\frac{\tilde{\zeta}_g + f}{\tilde{h}}\right) = 0 \quad \text{(C.45)}$$

The quantity inside the parentheses is the SWE potential vorticity, P_{SWE}.

The QG potential vorticity can be formed in an analogous way. Forming a vorticity equation from y derivative of (C.41) and an x derivative of (C.42) while using (C.19) and (C.23) yields the inviscid form of the vorticity equation in Cartesian, pressure coordinates:

$$\begin{aligned}&\frac{d_g\left(\zeta_g + f\right)}{dt} = \frac{\partial \zeta_g}{\partial t} + \vec{V}_g \bullet \vec{\nabla}\zeta_g + v_g\frac{\partial f}{\partial x_2} = \\ &\frac{\partial \zeta_g}{\partial t} + u_g\frac{\partial \zeta_g}{\partial x_1} + v_g\frac{\partial\left(\zeta_g + f\right)}{\partial x_2} = -f_o\left(\frac{\partial u}{\partial x} + \frac{1}{m_2}\frac{\partial v}{\partial y}\right) = f_o\frac{\partial \omega}{\partial p}\end{aligned} \quad \text{(C.46)}$$

Clearly this is in a similar form as (C.44) though the QG vorticity equation has vertical variation. A Cartesian, pressure coordinates form of (C.43) will have ω multiplying the static state static stability term. Eliminating ω from (C.43) and (C.46) yields:

$$\frac{dQ_{QG}}{dt} \equiv \frac{d}{dt}\left(\frac{1}{f_o}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left\{-\frac{f_o p}{RT_s \frac{\partial \ln\theta_s}{\partial p}}\frac{\partial\Phi}{\partial p}\right\}\right) = 0 \qquad \text{(C.47)}$$

where the quantity inside the brackets is the QG potential vorticity in this coordinate system (Holton, 2004). As with (C.45) this potential vorticity equation reveals a powerful constraint that the potential vorticity must be conserved following the flow (for frictionless, adiabatic motion). A height coordinate form (e.g. Vallis, 2006) is:

$$\frac{dQ_{QG}}{dt} \equiv \frac{d}{dt}\left(\frac{1}{f_o}\nabla^2 p + f + \frac{f_o}{\rho}\frac{\partial}{\partial z}\left\{-\frac{\rho}{N^2}\frac{\partial p}{\partial z}\right\}\right) = 0 \qquad \text{(C.48)}$$

where N^2 is the Brunt-Väisälä frequency.

The potential vorticity in isentropic coordinates also has a simple form. Combining (C.1) and (C.2) and using Kelvin's circulation theorem to a small loop on an isentropic surface recovers both the absolute vorticity and the area of that loop (Holton, 2004). For adiabatic motion a parcel's area on an isentropic surface swells or shrinks as adjacent isentropic surfaces approach or recede. For adiabatic motion:

$$\frac{dQ_E}{dt} \equiv \frac{d}{dt}\left(\left(\zeta_g + f\right)\left\{-\frac{g}{\frac{\partial p}{\partial\theta}}\right\}\right) = 0 \qquad \text{(C.49)}$$

Similar to (C.45) the term inside the curly brackets is always positive and inversely related to the spacing between θ surfaces. As the θ surfaces stretch apart, the absolute vorticity must decrease in magnitude to compensate, and *vice versa*.

C.4. Other useful relations

The area of a region on a sphere is a double integral of the area increment over the specified longitude λ and latitude φ ranges. The area increment is the longitude increment: r cosφ dλ multiplying the latitude increment: r dφ. Hence, when the full ranges of λ and φ are used, that double integral equals $4\pi r^2$. Hence:

$$\begin{aligned} &area\,of\ whole\,sphere = 4\pi r^2 = \int_{-\pi/2}^{\pi/2}\int_0^{2\pi} r^2\cos\varphi\, d\lambda\, d\varphi, \\ &area\,of\ region = \int_{\varphi_1}^{\varphi_2}\int_{\lambda_1}^{\lambda_2} r^2\cos\varphi\, d\lambda\, d\varphi \end{aligned} \qquad \text{(C.50)}$$